More generally. $y'' + a_1(x)y' + a_2(x)y = F(x)$

Suppose $y_1(x)$ is a homogeneous solution. We assume solution is

$$y = u(x)\, y_1(x) \quad \Rightarrow \quad \text{Find } u(x)$$

Plug into DE.

$$\begin{aligned} &(u''y_1 + 2u'y_1' + \overbrace{uy_1''}^{1,0}) \\ &+ a_1(x)\,(u'y_1 + uy_1') \qquad = F(x) \\ &+ a_2(x) \qquad uy_1 \end{aligned}$$

$$\cancel{u\,[y_1'' + a_1(x)y_1' + a_2(x)y]} + u''y_1 + u'[2y_1' + a_1(x)y_1] = F(x)$$

Put $w = u' \Rightarrow \quad w'y_1 + w\,[2y_1' + a_1(x)y_1] = F(x)$

1st order linear D.E. $\quad w' + \underbrace{\left[\frac{2y_1' + a_1(x)y_1}{y_1}\right]}_{p(x)} w = \underbrace{\frac{F(x)}{y_1}}_{q(x)}$

$$I(x) = e^{\int \left(\frac{2y_1'}{y_1} + a_1\right) dx} = y_1^2(x)\, e^{\int a_1(x)\,dx}$$

$$\frac{d}{dx}\left(I(x)\, w\right) = \frac{I(x)\,F(x)}{y_1(x)}$$

$$w(x) = \frac{1}{I(x)} \int^x \frac{I(s)\,F(s)}{y_1(s)}\,ds + \frac{C_1}{I(x)} = u'(x)$$

$$u(x) = \int^x \frac{1}{I(t)} \int^t \frac{I(s)\,F(s)}{y_1(s)}\,ds + C_1 \int^x \frac{1}{I(s)}\,ds + C_2$$

$$y = u(x)\,y_1 = C_2 y_1 + C_1 \underbrace{y_1 \int^x \frac{1}{I(s)}\,ds}_{y_2}$$

$$+ \underbrace{y_1 \int^x \frac{1}{I(t)} \int^t \frac{I(s)\,F(s)}{y_1(s)}\,ds}_{y_p}$$

Recall: the general solution to D.E. can be written as

$$y = C_1 y_1 + C_2 y_2 + y_p.$$

Remark: Memorize the algorithm, but not the final formula